

- Law of total probability
- Change order of summation
- Bayes theorem - conditionalize probabilities
- $X = 1 \Rightarrow P_X(s) = s$
- $P_X(0) = p_0$
- $P_X(1) = \sum p_k = 1$ if proper
- Find a recursion which gf of S satisfies:
 - S_i : population generated by i'th member of 1st generation
 - $S \sim^d S_i$

0.1 Generating Functions

- Sum of series: $\sum_{k=0}^n p^k = \frac{1-p^{n+1}}{1-p}$
- Random sums:

$$S_N = \sum_{i=1}^N X_i \Rightarrow P_{S_N}(s) = P_N(P_{X_1}(s))$$

- Gf of a convolution is product of gf's:

$$Y = X_1 + X_2 \Rightarrow P_Y(s) = P_{X_1}(s) \times P_{X_2}(s)$$

- Boundary condition: $P(X < \infty) = 1 \Rightarrow P_X(1) = 1$ proper distribution
- $\sum p_k s^k$ benzeri bir ifade gördüğünde bunu $P_X(s)$ ile değiştir
- Momentler:

$$p_k = 1/k! P_X^{(k)}(0)$$

$$E(X) = P'_X(s)|_{s=1} \text{Var}(X) = P_X^{(2)}(s)|_{s=1}$$

$$P_X(s) = E(s^X) = \sum_{k=0}^{\infty} p_k s^k$$

- Solve recursion with gf: olasılık dağılımı için kapalı bir ifade bul, gf'leri kullanarak. Yani bilinen gf'lere benzeyen bir ifade çıkart.
- Bilinen gf'ler:

$$P : e^{\lambda(s-1)} \quad B : (q + ps)^n \quad G : \frac{p}{1 - qs}$$

0.2 Branching Processes

- $P_n(s) = P_{n-1}(P(s))$
- $\pi = P(\pi)$ for $m > 1$, $\exists \pi < 1$
- $m = E(Z_1) \quad m_n = E(Z_n) = m^n$

0.3 Markov Chains

- $P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = P(X_{n+1} = j | X_n = i) = P_{ij}$
- recurrent state: chain returns to starting state wp 1 in finite steps: $P_i(\tau_i < \infty) = 1$
- transient state: o/w: $P_i(\tau_i = \infty) > 0, \{\tau_i = \infty\}$: never happens
- closed set of states: $C \subseteq S$ closed if $\forall i \in C: P_i(\tau_C = \infty) = 1 \quad i \in C \quad C^C$: comp of C
- τ_{C^C} : first entry to C comp. Yani hep C'de kalıyor.
- τ_j : first time of hitting state j.

$$\tau_B = \min\{n \geq 0; x_n \in B\}$$

- accessible: $i \rightarrow j$ if $P_i\{\tau_{au_j} < \infty\} > 0$, P_i : Pr.
starting from i
veya $i \rightarrow j$ iff $\exists n \geq 0 : P_{ij}^{(n)} > 0$
- communicate: $i \leftrightarrow j \Leftrightarrow i \rightarrow j \text{ and } j \rightarrow i$
- transitive: $i \leftrightarrow j, j \leftrightarrow k \Leftrightarrow i \leftrightarrow k$
- class: C_0 : set of all states communicating w. 0

$$\bigcup_i C_i = S \quad C_i \cap C_j = \emptyset$$

- irreducible MC: $\forall j \in S_i, i \leftrightarrow j$
- positive recurrent: E_i : number of steps to first hitting i starting from i. $E_i(\tau_i) < \infty$
- $f_{ij}^{(n)} = P_i\{\tau_j = n\}$: Pr of first visit to j from i in n steps
 $f_{ij}^{(n)} \neq P_{ij}^{(n)}$ in gen.
- $f_{ij} = \sum_{n=0}^{\infty} f_{ij}^{(n)} = P_i\{\tau_j < \infty\}$ Pr of ever visiting j from i. recurrent state iff $f_{ii} = 1$
- No of visits from i to j:

$$P_i\{N_j = m\} = \begin{cases} 1 - f_{ij} & ,m=0 \\ f_{ij}f_{jj}^{m-1}(1 - f_{jj}) & ,m=1,2,\dots \end{cases}$$

- if $f_{jj} < 1 \Rightarrow E_j(N_j) = \sum_{m=1}^{\infty} m f_{jj}^{m-1} (1 - f_{jj}) = \frac{1}{1-f_{jj}}$